

A better lower bound on average degree of 4-list-critical graphs

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Abstract

This short note proves that every incomplete k -list-critical graph has average degree at least $k - 1 + \frac{k-3}{k^2-2k+2}$. This improves the best known bound for $k = 4, 5, 6$. The same bound holds for online k -list-critical graphs.

1 Introduction

A graph G is *k -list-critical* if G is not $(k - 1)$ -choosable, but every proper subgraph of G is $(k - 1)$ -choosable. For further definitions and notation, see [5, 2]. Table 1 shows some history of lower bounds on the average degree of k -list-critical graphs.

Main Theorem. *Every incomplete k -list-critical graph has average degree at least*

$$k - 1 + \frac{k - 3}{k^2 - 2k + 2}.$$

Main Theorem gives a lower bound of $3 + \frac{1}{10}$ for 4-list-critical graphs. This is the first improvement over Gallai's bound of $3 + \frac{1}{13}$. The same proof shows that Main Theorem holds for online k -list-critical graphs as well. Our primary tool is a lemma proved with Kierstead [6] that generalizes a kernel technique of Kostochka and Yancey [8].

Definition. The *maximum independent cover number* of a graph G is the maximum $\text{mic}(G)$ of $\|I, V(G) \setminus I\|$ over all independent sets I of G .

Kernel Magic (Kierstead and R. [6]). *Every k -list-critical graph G satisfies*

$$2 \|G\| \geq (k - 2) |G| + \text{mic}(G) + 1.$$

The previous best bounds in Table 1 for k -list-critical graphs hold for k -Alon-Tarsi-critical graphs as well. Since Kernel Magic relies on the Kernel Lemma, our proof does not work for k -Alon-Tarsi-critical graphs. Any improvement over Gallai's bound of $3 + \frac{1}{13}$ for 4-Alon-Tarsi-critical graphs would be interesting.

	k -Critical G				k -List Critical G			
k	Gallai [4] $d(G) \geq$	Kriv [9] $d(G) \geq$	KS [7] $d(G) \geq$	KY [8] $d(G) \geq$	KS [7] $d(G) \geq$	KR [5] $d(G) \geq$	CR [2] $d(G) \geq$	Here $d(G) \geq$
4	3.0769	3.1429	—	3.3333	—	—	—	3.1000
5	4.0909	4.1429	—	4.5000	—	4.0984	4.1000	4.1176
6	5.0909	5.1304	5.0976	5.6000	—	5.1053	5.1076	5.1153
7	6.0870	6.1176	6.0990	6.6667	—	6.1149	6.1192	6.1081
8	7.0820	7.1064	7.0980	7.7143	—	7.1128	7.1167	7.1000
9	8.0769	8.0968	8.0959	8.7500	8.0838	8.1094	8.1130	8.0923
10	9.0722	9.0886	9.0932	9.7778	9.0793	9.1055	9.1088	9.0853
15	14.0541	14.0618	14.0785	14.8571	14.0610	14.0864	14.0884	14.0609
20	19.0428	19.0474	19.0666	19.8947	19.0490	19.0719	19.0733	19.0469

Table 1: History of lower bounds on the average degree $d(G)$ of k -critical and k -list-critical graphs G .

2 The Proof

The connected graphs in which each block is a complete graph or an odd cycle are called *Gallai trees*. Gallai [4] proved that in a k -critical graph, the vertices of degree $k - 1$ induce a disjoint union of Gallai trees. The same is true for k -list-critical graphs ([1, 3]). For a graph T and $k \in \mathbb{N}$, let $\beta_k(T)$ be the independence number of the subgraph of T induced on the vertices of degree $k - 1$. When k is defined in the context, put $\beta(T) := \beta_k(T)$.

Lemma 1. *If $k \geq 4$ and $T \neq K_k$ is a Gallai tree with maximum degree at most $k - 1$, then*

$$2||T|| \leq (k - 2)|T| + 2\beta(T).$$

Proof. Suppose the lemma is false and choose a counterexample T minimizing $|T|$. Plainly, T has more than one block. Let A be an endblock of T and let x be the unique cutvertex of T with $x \in V(A)$. Consider $T' := T - (V(A) \setminus \{x\})$. By minimality of $|T|$,

$$2||T|| - 2||A|| \leq (k - 2)(|T| + 1 - |A|) + 2\beta(T').$$

Since T is a counterexample, $2||A|| > (k - 2)(|A| - 1)$. So, if $k > 4$, then $A = K_{k-1}$ and if $k = 4$, then A is an odd cycle. In both cases, $d_T(x) = k - 1$. Consider $T^* := T - V(A)$. By minimality of $|T|$,

$$2||T|| - 2||A|| - 2 \leq (k - 2)(|T| - |A|) + 2\beta(T^*).$$

Since T is a counterexample, $2||A|| + 2 > (k - 2)|A| + 2(\beta(T) - \beta(T^*))$. In T^* , all of x 's neighbors have degree at most $k - 2$. But $d_T(x) = k - 1$, so some vertex in $\{x\} \cup N(x)$ is in a maximum independent set of degree $k - 1$ vertices in T . Hence $\beta(T^*) \leq \beta(T) - 1$, which gives

$$2||A|| > (k - 2)|A|,$$

a contradiction since $k \geq 4$. □

Proof of Main Theorem. Let $G \neq K_k$ be a k -list-critical graph. The theorem is trivially true if $k \leq 3$, so suppose $k \geq 4$. Let $\mathcal{L} \subseteq V(G)$ be the vertices with degree $k - 1$ and let $\mathcal{H} = V(G) \setminus \mathcal{L}$. Put $\|\mathcal{L}\| := \|G[\mathcal{L}]\|$ and $\|\mathcal{H}\| := \|G[\mathcal{H}]\|$. By Lemma 1,

$$2\|\mathcal{L}\| \leq (k-2)|\mathcal{L}| + 2\beta(\mathcal{L})$$

Hence,

$$\begin{aligned} 2\|G\| &= 2\|\mathcal{H}\| + 2\|\mathcal{H}, \mathcal{L}\| + 2\|\mathcal{L}\| \\ &= 2\|\mathcal{H}\| + 2((k-1)|\mathcal{L}| - 2\|\mathcal{L}\|) + 2\|\mathcal{L}\| \\ &= 2\|\mathcal{H}\| + 2(k-1)|\mathcal{L}| - 2\|\mathcal{L}\| \\ &\geq 2\|\mathcal{H}\| + k|\mathcal{L}| - 2\beta(\mathcal{L}), \end{aligned}$$

which is

$$\beta(\mathcal{L}) \geq \|\mathcal{H}\| + \frac{k}{2}|\mathcal{L}| - \|G\|. \quad (1)$$

Let M be the maximum of $\|I, V(G) \setminus I\|$ over all independent sets I of G with $I \subseteq \mathcal{H}$. Then

$$\text{mic}(G) \geq M + (k-1)\beta(\mathcal{L}).$$

Applying Kernel Magic and using (1) gives

$$\begin{aligned} 2\|G\| &\geq (k-2)|G| + M + (k-1)\beta(\mathcal{L}) + 1 \\ &\geq (k-2)|G| + M + (k-1)\left(\|\mathcal{H}\| + \frac{k}{2}|\mathcal{L}| - \|G\|\right) + 1 \\ &= (k-2)|G| + M + (k-1)\|\mathcal{H}\| + \frac{k(k-1)}{2}|\mathcal{L}| - (k-1)\|G\| + 1. \end{aligned}$$

Hence

$$(k+1)\|G\| \geq (k-2)|G| + M + (k-1)\|\mathcal{H}\| + \frac{k(k-1)}{2}|\mathcal{L}| + 1 \quad (2)$$

Let \mathcal{C} be the components of $G[\mathcal{H}]$. Then $\alpha(C) \geq \frac{|C|}{\chi(C)}$ for all $C \in \mathcal{C}$. Whence

$$M + (k-1)\|\mathcal{H}\| \geq \sum_{C \in \mathcal{C}} k \frac{|C|}{\chi(C)} + (k-1)\|C\|. \quad (3)$$

If $\mathcal{L} = \emptyset$, then G has average degree at least $k \geq k - 1 + \frac{k-3}{k^2-2k+2}$. So, assume $\mathcal{L} \neq \emptyset$. Then $G[\mathcal{H}]$ is $(k-1)$ -colorable by k -list-criticality of G . In particular, $\chi(C) \leq k-1$ for every $C \in \mathcal{C}$. For every $C \in \mathcal{C}$,

$$k \frac{|C|}{\chi(C)} + (k-1)\|C\| \geq \left(k - \frac{1}{2}\right)|C|. \quad (4)$$

To see this, first suppose $C \in \mathcal{C}$ is not a tree. Then $\|C\| \geq |C|$ and hence $k \frac{|C|}{\chi(C)} + (k-1)\|C\| \geq k \frac{|C|}{k-1} + (k-1)|C| \geq (k - \frac{1}{2})|C|$. If C is a tree, then $\chi(C) \leq 2$ and hence

$k \frac{|C|}{\chi(C)} + (k-1) \|C\| \geq k \frac{|C|}{2} + (k-1)(|C| - 1) \geq (k - \frac{1}{2}) |C|$ unless $|C| = 1$. This proves (4) since the bound is trivially satisfied when $|C| = 1$.

Now combining (2), (3) and (4) with the basic bound

$$|\mathcal{L}| \geq k |G| - 2 \|G\|,$$

gives

$$\begin{aligned} (k+1) \|G\| &\geq (k-2) |G| + \left(k - \frac{1}{2}\right) |\mathcal{H}| + \frac{k(k-1)}{2} |\mathcal{L}| + 1 \\ &= \left(2k - \frac{5}{2}\right) |G| + \frac{k^2 - 3k + 1}{2} |\mathcal{L}| + 1 \\ &\geq \left(2k - \frac{5}{2}\right) |G| + \frac{k^2 - 3k + 1}{2} (k |G| - 2 \|G\|) + 1. \end{aligned}$$

After some algebra, this becomes

$$2 \|G\| \geq \left(k - 1 + \frac{k-3}{k^2 - 2k + 2}\right) |G| + \frac{2}{k^2 - 2k + 2}.$$

That proves the theorem. \square

The right side of equation (4) in the above proof can be improved to $k |C|$ unless C is a K_2 where both vertices have degree k in G . If these K_2 's could be handled, the average degree bound would improve to $k - 1 + \frac{k-3}{(k-1)^2}$.

Conjecture. *Every incomplete (online) k -list-critical graph has average degree at least*

$$k - 1 + \frac{k-3}{(k-1)^2}.$$

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